Noise Induced Loss of Tracking in Systems With Saturating Actuators and Antiwindup

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This technical note is devoted to a recently discovered phenomenon that takes place in feedback systems with saturating actuators, proportional-integral (PI) control, and antiwindup. Namely, in such systems, measurement noise induces steady-state error in step tracking, which is incompatible with the standard error coefficients. We quantify this phenomenon using stochastic averaging theory and show that the noise induced loss of tracking occurs only if antiwindup is present. An indicator that predicts this phenomenon is derived, and a rule-of-thumb, based on this indicator, is formulated. An illustration using a digital printing device is provided. [DOI: 10.1115/1.4027163]

1 Introduction

PI controllers are widely used in industrial applications. In systems with saturating actuators, PI controllers often incorporate antiwindup in order to mitigate adverse effects of the saturation, e.g., sluggish transient, oscillatory behavior, sometimes instability [1–5]. In such systems, we have recently observed that measurement noise can induce steady-state tracking error. This is incompatible with the standard error coefficients of the PI controlled system [6] and implies a loss of tracking. As an illustration, Fig. 1 shows a block diagram of a simple feedback system with PI control, antiwindup and measurement noise, and Fig. 2 shows its step response. As one can see, the output of the system exhibits nonzero steady-state tracking error. This occurs when the steady-state output of the PI controller is near the saturation limit, and measurement noise acts through the controller to persistently trigger the antiwindup mechanism. If measurement noise is removed, the output tracks the reference without any steady-state error. We refer to this phenomenon as noise induced tracking error (NITE).

The goal of this technical note is to analyze the NITE phenomenon. In particular, we provide the conditions under which NITE occurs and quantify the tracking error as a function of system parameters.

Systems with saturating actuators and controller windup have been extensively studied in theory and in practice. A chronologi-

ical bibliography for systems with saturating actuators is given in Ref. [7] and more recent advances on the topics of stability, tracking, and disturbance rejection are covered in Refs. [8–10]. Practical strategies for mitigating the effect of integrator windup have also been widely studied [1–5]. In particular, Ref. [4] provides linear matrix inequality based algorithms for synthesizing antiwindups that improve both transient responses and stability. Steady-state tracking loss due to control saturation has been characterized in Ref. [10]. However, the noise induced tracking loss phenomenon has not been reported.

The contributions of this technical note are: (1) reporting the newly discovered NITE phenomenon; (2) quantifying this phenomenon using stochastic averaging; and (3) defining a dimensionless indicator that predicts the noise induced tracking error.

The outline of this technical note is as follows: the problem formulation is given in Sec. 2, the analysis is carried out in Sec. 3, and Sec. 4 defines the dimensionless indicator that predicts the loss of tracking. Section 5 illustrates the application of the indicator in a digital printing device, where the NITE phenomenon was originally discovered. Section 6 provides discussion on applicability of NITE analysis to an antiwindup scheme different from Fig. 1. Conclusions are given in Sec. 7. All proofs are included in the Appendix.

2 Problem Formulation

Consider the single input single output (SISO) system shown in Fig. 3, where \( P(s) \) is the plant, \( K_p, K_i, K_{AW} \) are, respectively, proportional gain, integral gain, and antiwindup gain, and \( C_i(s) \) is a strictly proper transfer function that represents controller dynamics in addition to PI control. The signals \( r, d, n, y, e \) are reference, disturbance, measurement noise, output, and tracking error, respectively, and \( \text{sat}^b(u) \) is the saturating actuator defined by

\[
\text{sat}^b(u) = \begin{cases} 
\alpha, & u < \alpha \\
\beta, & \beta > u \\
u, & \alpha \leq u \leq \beta 
\end{cases}
\]

with \( \alpha < 0 \) and \( \beta > 0 \).

We introduce the following assumptions:

(A1) The reference \( r \) and disturbance \( d \) are constants and satisfy

\[
x < \frac{r}{P(0)} - d < \beta
\]

where \( P(0) \) is the dc gain of the plant.

(A2) The measurement noise \( n(t) \) is a zero mean wide-sense stationary Gaussian process with standard deviation \( \sigma_n \) and the bandwidth of \( \sigma_n \), which is assumed to be much larger than the bandwidth of the feedback system.

(A3) For any \( r, d, \) and \( n(t) \) satisfying (A1) and (A2), the steady-state distribution of the output process \( y \) exists.

Assumption (A1) is introduced to exclude tracking loss due only to actuator saturation. Indeed, as it is shown in Ref. [10], the ranges of trackable \( r \) and rejectable \( d \) are limited in systems with saturating actuators. Inequalities (2) ensure that \( r \) and \( d \) are in the ranges, where no steady-state tracking loss occurs due to saturation, allowing us to investigate tracking loss due to the measurement noise and antiwindup. Assumption (A2) is introduced so that we explicitly consider measurement noise of large \( \sigma_n \). This is to ensure, as will be explained in Sec. 3, the applicability of stochastic averaging theory for analysis. Assumption (A3) is introduced for the following reason: Since the tracking error \( e \) is a random process, we quantify NITE by the expected value of the tracking error in the steady-state given by

\[
m_e = \lim_{t \to \infty} E[e(t)]
\]

and existence of \( m_e \) is guaranteed by (A3).
Under these assumptions, we address the following two problems:

- Provide conditions under which NITE occurs, i.e., $m_e \neq 0$.
- Determine the magnitude of $m_e$ as a function of the system parameters.

### 3 Analysis

#### 3.1 Application of Stochastic Averaging Theory

The analysis is based on the stochastic averaging theory described in Ref. [11], which consists of the following. Consider the system

$$\dot{x} = f(x, n_e(t))$$

where $x \in \mathbb{R}^r, f : \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^r$, and $n_e(t)$ is zero mean WSS Gaussian process with standard deviation of $\sigma_n$ and the bandwidth of $\omega_n$. The subscript $e$ is intended to parameterize the noise in such a manner that $\epsilon \rightarrow 0$ as $\omega_n \rightarrow \infty$. Then, for $\epsilon$ sufficiently small, the solution $x(t)$ of Eq. (4) can be approximated in probability by the solution of the averaged equation

$$\bar{x} = \bar{f}(\bar{x})$$

where $\bar{\epsilon} \in \mathbb{R}^r, f : \mathbb{R}^r \rightarrow \mathbb{R}^r$, and $\bar{f}$ is the conditional expected value of $f$ with respect to the distribution of $n_e(t)$, i.e.

$$\bar{f}(\bar{x}) = \mathbb{E}_{n_e}[f(\bar{x}, n_e(t))]$$

Since the bandwidth of the measurement noise is typically much larger than that of the closed-loop system, the behavior of Eq. (4) can be studied using the averaged system (5).

Applying stochastic averaging transforms the system of Fig. 3 to the averaged system of Fig. 4. Signals $\bar{y}, \bar{u}$, and $\bar{e}$ are output, control, and error in the averaged system, respectively, and the function $h_0^\beta(\bar{u}; K_p\sigma_n)$ is given by

$$h_0^\beta(\bar{u}; K_p\sigma_n) = \frac{\beta}{2} \left( \bar{u} - \frac{\beta}{2} \text{erf} \left( \frac{\bar{u} - \bar{x}}{\sqrt{2K_p\sigma_n}} \right) \right)$$

where

$$\text{erf} \left( \frac{2}{\sqrt{2\pi}} \frac{\bar{x}}{\sqrt{\sigma_n}} \right) = \frac{2}{\sqrt{\pi} \sigma_n} \int_{\bar{x}}^{\infty} \exp(-t^2) dt$$

The derivation of $h_0^\beta(\bar{u}; K_p\sigma_n)$ is provided in the Appendix.

Equations (9)–(12) imply that $h_0^\beta(\bar{u}; K_p\sigma_n)$ has the same saturation limits as $\text{sat}_0^\beta(\bar{u})$, and as $\sigma_n$ becomes small, it converges to the saturation nonlinearity. Comparisons of $h_0^\beta(\bar{u}; K_p\sigma_n)$ and $\text{sat}_0^\beta(\bar{u})$ are given in Fig. 5. One can see that $h_0^\beta(\bar{u}; K_p\sigma_n)$ is a "smoothed" saturation, and the degree of smoothing is determined by the product $K_p\sigma_n$.

Under (A1)–(A3), the results in Ref. [11] ensure that $m_e$ is well approximated by $\bar{e}_{ss}$, the steady-state value of $\bar{e}(t)$ of the averaged system, i.e.

$$\bar{e}_{ss} = \lim_{t \rightarrow \infty} \bar{e}(t)$$

Therefore, we study NITE of the system of Fig. 3 using the $\bar{e}_{ss}$ of Fig. 4.

To demonstrate the efficacy of this approach, the averaging theory is applied to the motivating example in Sec. 1. Output responses of the original system and the averaged system are obtained by numerical simulations. In the simulations of the
original system, random Gaussian numbers generated at each simulation sample step (0.002 s) are used as measurement noise with large \( \sigma_n \). Figure 6 shows the outputs of the original and the averaged systems, and the two are practically identical. Numerical calculations yield \( m_s = 0.1246 \) for the original system and \( \bar{e}_{ss} = 0.1252 \) for the averaged system, which shows high accuracy of the analysis method.

3.2 Analysis of NITE. The following result on \( \bar{e}_{ss} \) provides the conditions under which NITE occurs.

**Theorem 1.** Assume that \( K_i \neq 0 \) and for any \( r \) and \( d \) satisfying (A1) and (A2), the averaged system of Fig. 4 has asymptotically stable equilibrium. Then

\[
\bar{e}_{ss} = \frac{K_{AW}(\bar{u}_{ss} - h^0_a(\bar{u}_{ss}; K_p; \sigma_n))}{K_i}
\]

(14)

where \( K_p, K_i, K_{AW} \) are proportional, integral, antiwindup gains, respectively, and \( \bar{u}_{ss} \) is the steady-state value of \( \bar{u} \), i.e.

\[
\bar{u}_{ss} = \lim_{t \to \infty} \bar{u}(t)
\]

(15)

**Proof.** See Appendix.

Interpretations of Theorem 1 are as follows:

(1) In systems without antiwindup, measurement noise does not induce NITE. This is clear from Eq. (14) that \( \bar{e}_{ss} = 0 \) if \( K_{AW} = 0 \) regardless of other system parameters, such as proportional control gain, integral control gain, and noise standard deviation.

(2) In systems with antiwindup, measurement noise induces NITE in most cases. The level of NITE is determined by proportional control gain, integral control gain, antiwindup gain, noise standard deviation, and other plant parameters. This can be seen as follows: Equation (14) shows for \( K_{AW} \neq 0 \), \( \bar{e}_{ss} = 0 \) if and only if \( \bar{u}_{ss} = h^0_a(\bar{u}_{ss}; K_p; \sigma_n) \) = 0. However, as shown in Eq. (12), the latter takes place only if the combination of reference \( r \) and disturbance \( d \) yields the value \( \bar{u}_{ss} \) equal to the midpoint of lower and upper saturation limits. Because \( r \) and \( d \) are external inputs, in most cases, the term \( \bar{u}_{ss} - h^0_a(\bar{u}_{ss}; K_p; \sigma_n) \) is nonzero and leads to \( \bar{e}_{ss} \neq 0 \). The value of \( \bar{e}_{ss} \) is defined by Eq. (14).

(3) Regardless of antiwindup, NITE is never present without measurement noise. By using Eq. (11), it can be shown that \( \bar{u}_{ss} - h^0_a(\bar{u}_{ss}; K_p; \sigma_n) \) approaches zero as \( \sigma_n \) tends to zero. Thus, according to Eq. (14), \( \bar{e}_{ss} \to 0 \) as \( \sigma_n \to 0 \).

The result of Theorem 1 also implies that a tradeoff between steady-state error and transient performance exists in the design of antiwindup gain for the system of Fig. 4. No antiwindup yields \( \bar{e}_{ss} = 0 \), but can lead to the classic integral windup behavior and poor transient performance. On the other hand, a nonzero antiwindup gain leads to improved transient but results in \( \bar{e}_{ss} \neq 0 \). An optimal design of antiwindup and controller gains, however, is beyond the scope of this technical note.

To quantify \( \bar{e}_{ss} \) in terms of system parameters, one has to solve, for \( \bar{u}_{ss} \), a set of algebraic equations that describe the equilibrium of the averaged system. These equations are given in the proof of Theorem 1 (see Eq. (A3)). Then \( \bar{e}_{ss} \) can be quantified using Eq. (14). In a special case when \( P(s) \) has a pole in the origin, the algebraic equations simplify to a scalar equation, and in this case, \( \bar{e}_{ss} \) is proportional to \( K_{AW}/K_i \). This result can be formulated as follows:

**Corollary 1.** Assume that \( P(s) \) of Fig. 4 has a pole at the origin. Then, under the assumptions of Theorem 1

\[
\bar{e}_{ss} = \frac{K_{AW}(\bar{u}_{ss} + d)}{K_i}
\]

(16)

where \( \bar{u}_{ss} \) is the solution of

\[
d + h^0_a(\bar{u}_{ss}; K_p; \sigma_n) = 0
\]

(17)

**Proof.** See Appendix.

Because antiwindup is necessary in many practical applications to prevent undesirable transient behavior, we investigate the condition under which \( \bar{e}_{ss} \) is small with nonzero \( K_{AW} \). According to Eq. (14), \( \bar{e}_{ss} \) is small if \( \bar{u}_{ss} - h^0_a(\bar{u}_{ss}; K_p; \sigma_n) \) is small, i.e., \( \bar{u}_{ss} \) is in the region where \( \bar{u} \approx h^0_a(\bar{u}; K_p; \sigma_n) \). As an illustration of such region, Fig. 7 shows the function \( h^0_a(\bar{u}; K_p; \sigma_n) \) along with \( \text{sat}_\beta(\bar{u}) \). Clearly, there exists a region of \( \bar{u} \) where \( h^0_a(\bar{u}; K_p; \sigma_n) \) and \( \bar{u} \) are almost identical.
level, and Eq. (19) ensures that $u$ is far from the upper saturation level, and Eq. (19) ensures that $\dot{u}$ is far from the upper saturation level. For large $K_p\sigma_n$, this region may not exist, implying that NITE cannot be avoided. If it exists, and $u_{\text{ss}}$ of the system of Fig. 4 belongs to this region for given $r$ and $d$, then the steady-state behavior of the system is close to that with linear actuator, which implies $e_{\text{ss}} \approx 0$. This leads to an indicator for NITE.

### 4 Indicator for Noise Induced Tracking Error

Consider the following scenario: Assume, for given $r$, $d$, and $\sigma_n$, that we obtained $\dot{u}_{\text{ss}}$ by solving the set of algebraic equations that describe the equilibrium of Fig. 4. Assume further that this solution $\dot{u}_{\text{ss}}$ belongs to the region characterized by Eqs. (18) and (19), i.e.

$$\dot{u}_{\text{ss}} - x > 3K_p\sigma_n$$  \hspace{1cm} (18)

$$\beta - \dot{u}_{\text{ss}} > 3K_p\sigma_n$$  \hspace{1cm} (19)

Then, since the averaged system behaves similarly to the system with linear actuator near this equilibrium, $\dot{u}_{\text{ss}}$ should be close to the controller output at the equilibrium of the system with linear actuator. Let us denote this linear control output by $u_{\text{L,ss}}$. Then, the following inequalities hold for $u_{\text{L,ss}}$:

$$u_{\text{L,ss}} - x > 3K_p\sigma_n$$  \hspace{1cm} (20)

$$\beta - u_{\text{L,ss}} > 3K_p\sigma_n$$  \hspace{1cm} (21)

Therefore, $u_{\text{L,ss}}$ satisfying Eqs. (20) and (21) is an indication that $e_{\text{ss}}$ of Fig. 4 is small.

While evaluating Eqs. (20) and (21) requires obtaining $\dot{u}_{\text{ss}}$ from the system of Fig. 4, evaluating Eqs. (22) and (23) is rather straightforward as $u_{\text{ss}}$ is given by

$$u_{\text{L,ss}} = \lim_{\dot{u} \to 0} \left( \frac{C(s)}{1 + P(s)C(s)} r - \frac{P(s)C(s)}{1 + P(s)C(s)} d \right) = \frac{r}{P(0)} - d$$  \hspace{1cm} (24)

where $C(s) = K_p + (K_i/s) + C_0(s)$.

Since assumption (A1) ensures $x < u_{\text{L,ss}} < \beta$, inequalities (20) and (21) are equivalent to, respectively

$$\frac{3K_p\sigma_n}{u_{\text{L,ss}} - x} < 1$$  \hspace{1cm} (25)

$$\frac{3K_p\sigma_n}{\beta - u_{\text{L,ss}}} < 1$$  \hspace{1cm} (26)

This leads to the definition of the indicator $\Delta_u$ for NITE

$$\Delta_u := \max \left( \frac{3K_p\sigma_n}{u_{\text{L,ss}} - x}, \frac{3K_p\sigma_n}{\beta - u_{\text{L,ss}}} \right)$$  \hspace{1cm} (27)

Because the term $u_{\text{L,ss}}$ approximates the steady-state control value, the indicator $\Delta_u$ gives an indication whether $u_{\text{L,ss}}$ is in the linear range of the smoothed saturation or not as depicted in Fig. 7.

The indicator $\Delta_u < 1$ if and only if $u_{\text{L,ss}}$ is in the region described by Eqs. (18) and (19). Thus, we state the following rule-of-thumb for the system of Fig. 3:

**Rule-of-Thumb 1.** The NITE is practically zero if the indicator $\Delta_u < 1$; otherwise, the noise induced tracking error is present to the degree defined by Eq. (14).

An attractive feature of this indicator is that it can be computed without solving nonlinear system dynamics. Clearly, it is computed based only on the dc gain of the plant, proportional controller gain, saturation limits, the standard deviation of the measurement noise, and the sizes of the reference and the disturbance. For the case that $d$ is not precisely known, the indicator may be checked for the range of $d$ in order to correctly predict the presence of NITE.

### 5 Control System Design Using NITE Indicator

#### 5.1 Xerographic Toner Concentration (TC) Control

TC control is a common control loop used in digital printing [12,13]. It was in this context that the noise induced tracking error phenomenon was originally discovered. In the development stage of the xerographic process, TC control maintains the ratio of toner mass to carrier mass in order to achieve acceptable image quality. A block diagram of a typical TC control system is of the form of Fig. 3, where reference $r$ and output $y$ are the desired and the actual TC (in percent), respectively, disturbance $d$ represents toner removal rate from the system due to printing, and $n$ is the sensor noise. The plant and controller parameters are given below and are scaled values based on an actual printer:

$$P(s) = \frac{0.0322}{s}, \quad C_0(s) = 0, \quad K_p = 0.82, \quad K_i = 0.0065,$$

$$K_{\text{AW}} = 0.25, \quad \alpha = 0, \quad \beta = 0.66$$  \hspace{1cm} (28)

#### 5.2 Analysis of NITE in TC Control

The loss of tracking due to measurement noise was discovered under so-called “low area coverage condition” (see Ref. [14]) represented by $d = -0.011$ and a step change of TC target, $r$, from 4% TC to 5% TC. The standard deviation of the measurement noise is equal to 0.06% TC. Under this condition, calculating the indicator yields

$$\Delta_u = \frac{3K_p\sigma_n}{d} = 13.35$$  \hspace{1cm} (29)

which, because $\Delta_u > 1$, indicates the presence of NITE. Carrying out the calculation of $\dot{e}_{\text{ss}}$ according to Eq. (16) gives $\dot{e}_{\text{ss}} = -1.21$ which means that the output TC will fail to track the reference of 5% TC but instead settle at 6.21% TC. Simulated responses of the TC control system are shown in Fig. 8. Figure 8(a) shows that the output indeed exhibits the predicted behavior, failing to track 5% TC and, instead, approaching to 6.21% TC. Figure 8(b) shows the response for the case of no measurement noise, and the output tracks 5% TC without any error, as expected from PI controlled system. This clearly shows that it is the measurement noise that induced the tracking loss. Figure 8(c) shows the response for the case where measurement noise is present but antwindup is not ($K_{\text{AW}} = 0$). As predicted in Theorem 1, the system no longer exhibits the NITE behavior, although the transient response is poor.

The relation between level of noise and level of NITE, obtained by using Theorem 1, is provided in Table 1. Clearly, measurement...
noise with higher standard deviation yields a higher level of NITE. As explained in Sec. 3.1, this is because a higher noise standard deviation yields a higher degree of smoothing in $h_b(u; K_p, \sigma_n)$, and in turn, yields a higher amplitude of the term $\mu_u - h_b(u; K_p, \sigma_n)$ in the expression of NITE given in Theorem 1.

5.3 Control System Redesign to Prevent NITE. To further illustrate the efficacy of the indicator, assume now that we reduce the proportional gain to $K_p = 0.61$ and switch to a higher quality sensor with less noise represented by $\sigma_n = 0.006\%$ TC. Then, recomputing Eq. (29) gives $\Lambda_n = 0.99$, and we would expect no NITE. Indeed, carrying out the TC control simulation results in no tracking error as shown in Fig. 9. Note that the overshoot and settling time are not as good as those of Fig. 8(b).

6 NITE in the Antiwindup of Ref. [5]

Although analysis in this paper is carried out for the system of Fig. 3, it turns out that the phenomenon of NITE occurs in feedback control systems with different antiwindups as well. We illustrate this using the antiwindup given in Ref. [5], where a PI type controller is split into the direct feedthrough term and the strictly proper term followed by feedback form implementation around the saturating actuator. Since no additional controller parameter is introduced, the method of Ref. [5] can be viewed as a controller implementation method that prevents windup rather than augmentation of antiwindup compensator.

Figure 10 shows the response of the system similar to the system of Fig. 1 where the plant transfer function is $(1/3s + 1)$ and PI controller with $K_p = 5$ and $K_i = 3$ is implemented in the manner shown in Ref. [5]. Clearly, with measurement noise, output of the system loses tracking and exhibits NITE. It is noted that NITE in Fig. 10 is smaller in magnitude when compared to that in Fig. 2. Nevertheless, it confirms that NITE occurs in feedback systems where controller is implemented as in Ref. [5] for windup prevention. Further analysis remains as future work.

7 Conclusions

For a system under PI-type control with actuator saturation and antiwindup, we have shown that measurement noise can induce a tracking loss resulting in a nonzero steady-state tracking error. This previously unreported phenomenon is referred to as NITE and was first observed in a TC control application, a common process control loop in digital printing. By applying stochastic averaging theory, we have derived an analytical expression of tracking error and shown how the antiwindup and sensor noise affect the loss of tracking. A simple dimensionless indicator has also been
derived and used to formulate the following rule-of-thumb: when $\Delta_u < 1$, NITE is absent and when $\Delta_u \geq 1$, NITE is present.

Generalizing these results to other forms of antiwindup and the design of antiwindup with a consideration of the tradeoff between steady state and transient performance are open questions.

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Appendix

Derivation of Eq. (7)
The function $h^R(u; K_p \sigma_a)$ is defined as the conditional expected value of sat$(u)$ with respect to Gaussian noise $n$

$$h^R(u; K_p \sigma_a) = \mathbb{E}_u[\text{sat}(u)] = \int_{-\infty}^{\infty} \text{sat}(u - K_p n) \frac{1}{\sqrt{2\pi\sigma_n}} e^{-\frac{u^2}{2\sigma_n^2}} dn$$  \hspace{1cm} (A1)

where $u$ is decomposed into $u = v - K_p n$ with $v$ representing the portion of $u$ not directly depending on $n$. Evaluating the integral in Eq. (A1) gives Eq. (7).

Proof of Theorem 1

Let a state space realization of the system of Fig. 4 be

$$\dot{x}_p = A_p x + B_p d + B_p h^R(u; K_p \sigma_a),$$

$$\dot{x}_c = A_{C_p} x_C + B_{C_p} (r - y),$$

$$\dot{y} = K_p (r - \bar{y}) + K_A W (h^R(u; K_p \sigma_a) - \bar{u}),$$

$$\dot{\bar{y}} = C_p \bar{x}_p,$$

$$\bar{u} = K_p (r - \bar{y}) + C_{C_p} x_C + \bar{x}$$

where $\bar{x}_p$ is the state of the plant, $x_C$ is the state of the additional controller dynamics, and $\bar{y}_i$ is the state of the integral controller.

The steady state of this system is defined by

$$A_{p} x + B_{p} d + B_{p} h^R(u; K_p \sigma_a) = 0,$$

$$A_{C_p} x_C + B_{C_p} (r - \bar{y}) = 0,$$

$$K_p (r - \bar{y}) + K_A W (h^R(u; K_p \sigma_a) - \bar{u}) = 0$$

where $\bar{u}$ and $\bar{y}$ are defined as in Eq. (A2).

Since the existence of asymptotically stable equilibrium is assumed

$$K_i (r - \bar{y}) + K_A W (h^R(u; K_p \sigma_a) - \bar{u}) = 0$$  \hspace{1cm} (A4)

must be satisfied in the steady state, where $\bar{y}_s$ and $\bar{u}_s$ are the steady-state values of $\bar{y}$ and $\bar{u}$. Therefore

$$\bar{e}_s = r - \bar{y}_s = \frac{K_A W (\bar{u}_s - h^R(\bar{u}_s; K_p \sigma_a))}{K_i}$$  \hspace{1cm} (A5)

Proof of Corollary 1

Since $P(s)$ has a pole at the origin, for an equilibrium to exist, the input to the plant $P(s)$ must be zero in the steady state, i.e.

$$d + h^R(\bar{u}_s; K_p \sigma_a) = 0$$  \hspace{1cm} (A6)

Therefore, using Theorem 1, we obtain

$$\bar{e}_s = \frac{K_A W (\bar{u}_s + d)}{K_i}$$  \hspace{1cm} (A7)

where $\bar{u}_s$ is the solution of Eq. (A6).

References


